International Journal of Advanced Research and Multidisciplinary Trends (IJARMT)
An International Open Access, Peer-Reviewed Refereed Journal
Impact Factor: 6.4 Website: https://ijarmt.com ISSN No.: 3048-9458

Certain Denouement in Complete Metric Space with Application of Fixed-Point Theory

S.K. Jain
Professor, Dept. of Applied Mathematics, Ujjain Engineering College Ujjain (M.P.) India
Shoyeb Ali Sayyed
Professor, Dept. of Mathematics, Malwanchal University, Indore (M.P.) India

ABSTRACT

In this article, Author designed to reveal some developments in complete metric space with application of fixed-point theory with conceptions of number of great and renowned mathematicians. Applying square inequality and satisfies many results in complete metric space.

Keywords and Phrases: complete metric space, square inequality, continuous selfmaps, fixed point theory.

AMS (2010) SUBJECT CLASSIFICATIONS: Primary 47H10 Secondary 54H25,34B15

1. INTRODUCTION

The number of complications in pure, applied and industrial mathematics have an ideal explanation by fixed point theory. Consequently, numerous of strategy in functional analysis, numerical analysis, topology, number theory and approximation theory for achieving successive approximations to the fixed point of an approximate map.

The familiar Banach Contraction Principal Ankushrao and Sayyed [1] was nearly new by Jain and Sayyed [6], Dolhare et.al [3], Ankushrao and Sayyed [2], Jachymaki [5], Gornicki [4], Liu, et.al. [8], Jungck, [7], Sayyed Et. al. [11], Sayyed and Badshah [12], Mane and Sayyed [9], Murthy, P.P., Vara Prasad, K.N.V.V and Rashmi [10], Sayyed [12], Vyas and Sayyed [14], Yang [15] , Suhas and Dolhare[13] and many others.

2.PRELIMINARY

Main result: If X is a complete metric space and R is a continuous self map on X and satisfies the following condition.

$$[d(Rx, Ry)]^2 \leq a_1 [d(Rx, x)\, d(Ry, y)+d(Ry,x)d(Rx,y)]+a_2 [d(Rx,x)d(Ry,x)+d(Rx,Ry)d(Rx,y)] + a_3 [d(x,y)d(Rx,x)+d(Ry,x)d(Rx,y)]$$

for all $x, y \in X$, $x \neq y$ and $a_1, a_2, a_3 \in [0,1]$ and $a_1 + 2a_2 + a_3 \leq 1$, then R has a unique fixed point.

International Journal of Advanced Research and Multidisciplinary Trends (IJARMT)
An International Open Access, Peer-Reviewed Refereed Journal
Impact Factor: 6.4 Website: https://ijarmt.com ISSN No.: 3048-9458

Proof: Let x_0 be an arbitrary point X, and we define a sequence $\{x_n\}$ by means of iterates of R. By setting, $R^n x_0 = x_n$, where n is a positive integer. If $x_n = x_{n+1}$, for some n, then we have $Rx_n = x_n$, then x_n is a fixed point of R taking $x_n \neq x_{n+1}$ for all n.

Now, $[d(x_{n+1}, x_n)]^2 = [d(Rx_n, Rx_{n-1})]^2$

$\leq a_1 [d(Rx_n, x_n) d(Rx_n, x_{n-1}) + d(Rx_{n-1}, x_n) d(Rx_{n-1}, x_n) d(Rx_n, x_{n-1})] + a_2 [d(Rx_n, x_n) d(Rx_{n-1}, x_n) + d(Rx_n, Rx_{n-1}) d(Rx_n, x_{n-1})] + a_3 [d(x_n, x_{n-1}) d(Rx_n, x_n) + d(Rx_{n-1}, x_n) d(Rx_n, x_{n-1})]$.

i.e. $d(x_{n+1}, x_n) \leq \frac{a_1 + a_2 + a_3}{1 - a_2} d(x_n, x_{n-1})$

Applying the same process, we get

. $d(x_{n+1}, x_n) \leq \frac{a_1 + a_2 + a_3}{1 - a_2} d(x_0, x_1)$

for m>n we have

$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)$

$\leq (k^n + k^{n+1} + \dots + k^{m-1}) d(x_0, Rx_0)$ where $k = \frac{a_1 + a_2 + a_3}{1 - a_2} < 1$

Therefore $d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, Rx_0) \to 0$ as $m, n \to \infty$. So $\{x_n\}$ is a cauchy sequence in X, so by completeness there is a point $l_1 \in X$ such that $x_n \to l_1$, as $n \to \infty$. The continuous of R implies $R(l_1) = R(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} x_{n+1} = l_1$, Therefore l_1 is a fixed point of R.

Uniqueness: If any other $l_1 \neq l_2$ in X, such that $R(l_2) = l_2$, then $[d(l_1, l_2)]^2 = [d(Rl_1, Rl_2)]^2$

$\leq a_1[d(Rl_1, l_1) d(Rl_2, l_2) + d(Rl_2, l_1) d(Rl_1, l_2)] + a_2 [d(Rl_1, l_1) d(Rl_2, l_2) + d(Rl_1, Rl_2) d(Rl_1, l_2)] + a_3 [d(l_1, l_2) d(Rl_1, l_1) + d(Rl_2, l_1) d(Rl_1, l_2)]$

i.e. $d(l_1, l_2) \leq (a_1 + a_2 + a_3) d(l_1, l_2)$

which is the contradiction. Hence l_1 is a unique fixed point. Now we are proving on interesting result in which R is not necessarily continuous in X, but R^p is continuous for some positive integer p, R^p is continuous then R has a unique fixed point.

Theorem 2: Let R be a self map defined on a complete metric space (X, d) such that (1:1) holds. If for some positive integer P, R^p is continuous, then R has a unique fixed point.

PROOF: We define a sequence $\{x_n\}$ as in theorem 1, clearly it converges to same point l_1 in X. Therefore, there is a subsequence $\{x_{nk}\}$ of $\{x_n\}$, $(n_k = k_p)$ also converges to l_1. Also

$$R^p l_1 = R^p(\lim_{k\to\infty} x_{nk})$$

$$= \lim_{k\to\infty} (R^p x_{nk})$$

$$= \lim_{k\to\infty} (x_{nk+1})$$

$= l_1$.

Therefore l_1 is a fixed point of R^p. Now, we show that $Rl_1=l_1$.

Let m be the smallest positive integer such that $R^m l_1 = l_1$, But $R^q l_1 \neq l_1$. For q = 1,2,3,....., m-1, if m>1

$[d(Rl_1, l_1)]^2 = [d(Rl_1, R^m l_1)]^2$

$=[d(Rl_1, R(R^{m-1} l_1))]^2$

$\leq a_1 [d(Rl_1, l_1) d(R^m l_1, R^{m-1}l_1) + d(R^m l_1, l_1) d(Rl_1,R^{m-1}l_1)] + a_2 [d(Rl_1,l_1) d(R^m l_1,l_1) + d(Rl_1, R^m l_1)$
$d(Rl_1,R^{m-1} l_1)] + a_3 [d(l_1, R^{m-1}l_1)d(Rl_1, l_1) + d(R^m l_1, l_1) d(Rl_1, R^{m-1}l_1)$

$\leq a_1 d(Rl_1,l_1) d(l_1, R^{m-1}l_1) + a_2 d(Rl_1,l_1) d(Rl_1, R^{m-1}l_1) + a_3 d(l_1, R^{m-1}l_1) d(Rl_1,l_1)$

i.e. $d(Rl_1,l_1) \leq \frac{a_1+a_2+a_3}{1-a_2} d(l_1, R^{m-1}l_1)$

or

$d(Rl_1,l_1) \leq k^m d(l_1, R^{m-1}l_1)$, thus we can write

$d(Rl_1,l_1) \leq k^m d(l_1, Rl_1)$, since $k^m < 1$, therefore

$d(Rl_1,l_1) < d(Rl_1,l_1)$, which is a contradiction. Hence l_1 is a fixed point of R.

The uniqueness of l_1 follows as in theorem 1.

Theorem 3. Let R be a continuous self map of X. where X is complete metric space such that for some h > 0 then,

$$[d(R^h x,R^h y)]^2 \leq a_1 [d(R^h x,x)d(R^h y,y) + d(R^h y,x)d(R^h x,y)] + a_2 [d(R^h x,x) d(R^h y,x)+d(R^h x,R^h y)d(R^h x,y)] + a_3 [d(x, y) d(R^h x,x)+ d(R^h y,x)d(R^h x,y)]$$

for all x,y $\in$ X and a_1, a_2 and $a_3 > 0$ with $a_1 + 2a_2 + a_3 \leq 1$, If R^h is continuous , then R has a unique fixed point.

Proof. According to theorem 2 we assume that R^h has a unique fixed point, also

$Rl_1 = R(R^h l_1) = R^h(Rl_1)$.

which implies $Rl_1 = l_1$. Since fixed point of R is a fixed point of R^h and R^h has a unique fixed point l_1 . It follows that l_1 is the unique fixed point of R.

Conclusion

This study establishes a certain denouement in a complete metric space through the application of fixed-point theory. By formulating appropriate contractive-type conditions, the existence and uniqueness of a fixed point are guaranteed under completeness assumptions. The results demonstrate that completeness of the metric space plays a crucial role in ensuring convergence of iterative sequences generated by the given mappings. Furthermore, the proposed framework unifies and extends several classical fixed-point results as special cases. These findings not only strengthen the theoretical foundations of fixed-point theory but also enhance its applicability to nonlinear analysis, differential equations, and optimization problems where solution certainty and convergence assurance are essential.

References

[1] Ankushrao, M.A. and Sayyed, S.A. (2022) A review of exploring applications of fixed-point theory across mathematical spaces and equations, International Journal of Food and Nutritional Sciences, Vol. 11, Iss 13, 1356-1364.

[2] Ankushrao, M.A. and Sayyed, S.A, (2023) Exploring applications of fixed-point theory across mathematical spaces and equations, Monthly Baraheen, Vol. 15, Iss 177, 2201-2209,

[3] Dolhare U.P. and Khanpate V.B., (2016), Generalized Study of Commuting Self-Maps and Fixed Points, Int.Jou.Sci.& Res., Vol.5(12), 648-651.

[4] Gornicki, J., (2017) Fixed point theorem for Kannan type mappings. J. Fixed Point Theory Appl.19, 2145–2152.

[5] Jachymski, J., (1995) Fixed point theorems for expansive mappings, Math. Japonica, 42, No.1, 131-136.

[6] Jain, S.K. and Sayyed, S.A, (2019) " Nonlinear contraction satisfying square inequality and fixed point theory", International Journal of Research and Analytical Reviews , Vol. 6, Issue 2,923-2929.

[7] Jungck, G., (1986), Compatible mappings and common fixed points, Int. J. Math. Sci. 9(4) 771-779.

[8] Liu, W., Wu, J. and Li, Z., (2005), Common fixed points of single valued and multivalued maps, Int. J Math. Sci., 19, 3045-3055.

[9] Mane, S.P. and Sayyed, S.A. (2021) Study on contractive maps with different types of fixed points, The Journal of Oriental Research Madras, Vol. MMXXII-XCIII-II, 70-84.

[10] Murthy, P.P., Vara Prasad, K.N.V.V and Rashmi, (2013), Fixed points of nonlinear contraction, Adv. Fixed point Theory, 3 No. 4, 600-607.

[11] Sayyed, S.A., Devghare, K. and Badshah, V.H., *(2006)* A Note on Fixed Point for Selfmaps", Acta *Ciencia Indica", Vol. XXXII M, No. 4, 1595-1596.*

[12] Sayyed, S.A. and Jain, S.K. (2022), Pairs of Compatible Type (B) Mappings and Fixed-Point Theory in Banach Space, Journal of Mathematical Sciences & Computational Mathematics, Vol.4, No1, 87-96.

[13] Suhas, P., and Dolhare U.P., (2016), A Note on Development of Metric fixed-point Theory. International Journal of Advanced Research Res.4(8), vol. 4(31).

[14] Vyas, L. And Sayyed, S.A., (2014) "Common Fixed Points For Multivalued Mappings" , Advances In Applied Mathematical Biosciences,Vol.5 No.119-25 .

[15] Yang, Z., (2016) Common fixed point for three pairs of slf maps satisfying weakly commuting and weakly compatible condition in generalized metric space, Journal of Nonlinear Science and Applications, Vol.9,3962-3979.