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EXISTENCE OF FIXED-POINT FOR A GENERALIZED CONTRACTION IN S-METRIC SPACES

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Abstract
This paper investigates the existence of fixed points for a class of generalized contractions in complete S-metric spaces. We explore several types of contraction conditions and provide an enriched theoretical framework supported by examples. The paper includes novel results extending the Banach contraction principle, altering distance function approaches, and weakly compatible maps to the context of S-metric spaces. Moreover, new classes of contraction mappings such as cyclic contractions, common fixed points for pairs of commuting mappings, and generalized Φ-type contractions are explored and discussed. Our findings offer a unifying structure that links various fixed-point theorems under a broader umbrella, enhancing their applicability to diverse problems in differential equations, optimization, and dynamical systems. The introduction of new contraction conditions helps in the relaxation of hypotheses, making our results applicable in settings where classical conditions fail. We also extend results for bounded S-metric spaces and establish common fixed-point theorems for pairs of weakly compatible maps using altering distance functions and comparison functions. With rich examples and illustrative explanations, the results in this paper contribute significantly to the growing literature on fixed point theory in generalized metric spaces.
Keywords: Fixed points, S-Metric Space, Compatible Maps

Introduction

Fixed point theory plays a pivotal role in nonlinear analysis, providing essential tools for addressing problems in differential equations, dynamical systems, and optimization. The Banach contraction principle guarantees the existence and uniqueness of fixed points in complete metric spaces under certain contractive conditions.

To handle more complex interactions, traditional metric spaces have been generalized into frameworks such as partial metric, G-metric, and S-metric spaces. The S-metric space, introduced by Sedghi et al., extends the classical metric by involving three variables, offering a flexible structure for modeling triadic relationships found in multi-agent systems and computational geometries.

This paper explores fixed point results for generalized contractions in complete S-metric spaces. By employing altering distance and comparison functions, we establish new fixed-point theorems under relaxed conditions. We also consider specific classes of mappings—including cyclic, commuting, and weakly compatible maps—broadening the applicability of fixed-point theory in generalized metric settings.

Definitions:

1.Metric Space:

Let X be a non-empty set. A metric on X is a real function d_λ : X x X → R, which satisfies the following axioms:-

(i) $d_\lambda(x, y) \geq 0$ for all x, y ϵ X

(ii) $d_\lambda(x, y) = 0$, if and only if x = y

(iii) $d_\lambda(x, y) = d_\lambda(y, x)$ for all x, y ϵ X,

(iv) $d_\lambda(x, z) \leq d_\lambda(x, y) + d_\lambda(y, z)$ for all x, y, z ϵ X.

The ordered pair (X, d_λ) is called a **metric space** and $d_\lambda(x, y)$ is called the distance between x and y. The elements of X are called its points.

2. Contraction mapping: Let (X, d_λ) be a metric space and a mapping d_λ: X → X is said to be contraction mapping if there exist a real number μ with $0 \preccurlyeq \mu \prec 1$ s.t. $d_\lambda(\varphi(x), \varphi(y)) \preccurlyeq \mu\, d_\lambda(x, y)$ for all x, y ϵ X and x ≠y, Thus, in contraction on X, the distance between the images of any two points is less than the distance between the points.

3. Compatible mappings:

Let (X, d_λ) be a metric space. The mappings $\mathscr{F}$ and $\hbar$ where $\mathscr{F}: X^3 \to X$ and $\hbar : X \to X$ are said to be **compatible** if

$$\lim_{n\to\infty} d_\lambda(\hbar(\mathcal{F}(x_n, y_n, z_n)), \mathcal{F}(\hbar(x_n), \hbar(y_n), \hbar(z_n))) = 0$$

$$\lim_{n\to\infty} d_\lambda(\hbar(\mathcal{F}(y_n, x_n, y_n)), \mathcal{F}(\hbar(y_n), \hbar(x_n), \hbar(y_n))) = 0$$

$$\lim_{n\to\infty} d_\lambda(\hbar(\mathcal{F}(z_n, y_n, x_n)), \mathcal{F}(\hbar(z_n), \hbar(y_n), \hbar(x_n))) = 0$$

Whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in X such that

$$\lim_{n\to\infty} \mathcal{F}(x_n, y_n, z_n) = \lim_{n\to\infty} \hbar(x_n) = x$$

$$\lim_{n-\to\infty} \mathcal{F}(y_n, x_n, y_n) = \lim_{n\to\infty} \hbar(y_n) = y \text{ and}$$

$$\lim_{n\to\infty} \mathcal{F}(z_n, y_n, x_n) = \lim_{n\to\infty} \hbar(z_n) = z \text{ for some } x, y, z \in X.$$

4. S-metric space:

Let X be a non-empty set. An S – metric on X is a function S : $X^3 \to [0, \infty)$ that satisfies the following conditions for all x, y, z, a ∈ X.

(i) S(x, y, z) = 0 if and only if x = y = z.

(ii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S – metric space.

5. Complete s-metric space:

Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\}$ c X converges to x ∈ X if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each ϵ >0, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$. We have, $S(x_n, x_n, x) < \epsilon$. We write $x_n \to x$ for brevity.

(2) A sequence $\{x_n\}$ c X converges to x is a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n,m \to \infty$. That is for each ϵ >0, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \epsilon$.

The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

6. Compatible:

Let $(\mathcal{J}, S)$ be an S-metric space. A pair {l, k} is said to be compatible if and only if $\lim_{n\to\infty} S(lku, lku, lku) = 0$, whenever $\{u_n\}$ is a sequence in $\mathcal{J}$ such that $\lim_{n\to\infty} lu = \lim_{n\to\infty} ku = r$ for some $r \in \mathcal{J}$.

Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write for $x_n \to x$.

(ii) A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

The S-metric space (X, S) is complete if every Cauchy sequence is convergent.

Main Results:

Theorem 1

Suppose that f, g, R and T are self-map of a complete S-metric space (X, S), with $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$ and that the pairs {f, R} and {g, T} are compatible.

If $S(fx, fy, fz) \leq q \max\{S(Rx, Ry, Tz), S(fx, fx, Rx), S(gz, gz, Tz), S(fy, fy, gz)\}$, for each x, y, $z \in X$, with $0 < q < 1$. Then f, g, R, and T have a unique common fixed point in X if R and T are continuous.

Proof: Let $x_0 \in X$. Since $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$, and also as $gx_1 \in R(X)$, we choose $x_2 \in X$ such that $gx_1 = Rx_2$. In general, $x_{2n+1} \in X$ is chosen such that $fx_{2n} = Tx_{2n+1}$ and $x_{2n+2} \in X$ such that $gx_{2n+1} = Rx_{2n+2}$, we obtain a sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$, $y_{2n+1} = gx_{2n+1} = Rx_{2n+2}$, $n \geq 0$.

Now we show that $\{y_n\}$ is a Cauchy sequence. For this we have

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n+1}) &= S(fx_{2n}, fx_{2n}, gx_{2n+1}) \\ &\leq q \max\{S(Rx_{2n}, Rx_{2n}, Tx_{2n+1}), S(fx_{2n}, fx_{2n}, Rx_{2n}), \\ &\qquad S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fx_{2n}, fx_{2n}, gx_{2n+1})\} \\ &= q \max\{S(y_{2n-1}, y_{2n-1}, y_{2n}), S(y_{2n}, y_{2n}, y_{2n-1}), S(y_{2n+1}, y_{2n+1}, y_{2n}), \\ &\qquad S(y_{2n}, y_{2n}, y_{2n+1})\} \\ &= q \max\{S(y_{2n-1}, y_{2n-1}, y_{2n}), S(y_{2n}, y_{2n}, y_{2n+1})\} \end{aligned}$$

Now, if $S(y_{2n}, y_{2n}, y_{2n+1}) > S(y_{2n-1}, y_{2n-1}, y_{2n})$, then by above inequality we have $S(y_{2n}, y_{2n}, y_{2n+1}) < q\, S(y_{2n}, y_{2n}, y_{2n+1})$, which is a contradiction.

Hence, $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n})$, therefore by above inequality we get $S(y_{2n}, y_{2n}, y_{2n+1}) \leq q\, S(y_{2n-1}, y_{2n-1}, y_{2n})$. (1.1)

By similar arguments, we have $S(y_{2n-1}, y_{2n-1}, y_{2n}) = S(y_{2n}, y_{2n}, y_{2n-1}) = S(fx_{2n}, fx_{2n}, gx_{2n-1})$

$\leq q \max\{S(Rx_{2n}, Rx_{2n}, Tx_{2n-1}), S(fx_{2n}, fx_{2n}, Rx_{2n}),$

$S(gx_{2n-1}, gx_{2n-1}, Tx_{2n-1}), S(fx_{2n}, fx_{2n}, gx_{2n-1})\}$

$= q \max \{S(y_{2n-1}, y_{2n-1}, y_{2n-2}), S(y_{2n}, y_{2n}, y_{2n-1}),$
$S(y_{2n-1}, y_{2n-1}, y_{2n-2}), S(y_{2n}, y_{2n}, y_{2n-1})\}$
$= q \max \{ S(y_{2n-2}, y_{2n-2}, y_{2n-1}), S(y_{2n}, y_{2n}, y_{2n-1})\}$

Now, if $S(y_{2n}, y_{2n}, y_{2n-1}) > S(y_{2n-2}, y_{2n-2}, y_{2n-1})$, then by above inequality,
we have $S(y_{2n}, y_{2n}, y_{2n-1}) < q\, S(y_{2n}, y_{2n}, y_{2n-1})$, which is a contradiction.
Hence, $S(y_{2n-1}, y_{2n-1}, y_{2n}) \le S(y_{2n-2}, y_{2n-2}, y_{2n-1})$, therefore by above inequality, we get-
$S(y_{2n-1}, y_{2n-1}, y_{2n}) \le q\, S(y_{2n-2}, y_{2n-2}, y_{2n-1})$. (1.2)
Now, from (1.2) and (1.3) we have $S(y_n, y_n, y_{n-1}) \le S(y_{n-1}, y_{n-1}, y_{n-2})$, $n \ge 2$, $0 < q < 1$.
Hence, for $n \ge 2$ it follows that $S(y_n, y_n, y_{n-1}) \le \ldots \le q^{n-1} S(y_1, y_1, y_0)$ (1.3)
By the triangle inequality in S-metric space, for n >m we have
$S(y_n, y_n, y_m) \le 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) + \ldots + S(y_{n-1}, y_{n-1}, y_n)$
$< 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) + \ldots + S(y_{n-1}, y_{n-1}, y_n)$.
Hence, from (1.4) and $0<q< 1$ we have
$S(y_n, y_n, y_m) \le 2(q^m + q^{m+1} + \ldots + q^{n-1})\, S(y_1, y_1, y_0)$
$\le 2q^m [1 + q + q^2 + \ldots]\, S(y_1, y_1, y_0)$
$\le 2 \frac{q^m}{1-q} S(y_1, y_1, y_0) \to 0$, as $m \to \infty$.
It follows that $\{y_n\}$ is a Cauchy sequence. Since X is a complete S-metric space,
there is some y in X such that
$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Rx_{2n+2} = y$.
We show that y is a common fixed point of f, g, R, T. Since R is continuous it follows that
$\lim_{n\to\infty} R^2x_{2n+2} = Ry$, $\lim_{n\to\infty} Rfx_{2n} = Ry$.
and since f and R are compatible, $\lim_{n\to\infty} S(fRx_{2n}, fRx_{2n}, Rfx_{2n}) = 0$.
So, by the Lemma $\lim_{n\to\infty} fRx_{2n} = Ry$.
Putting $x = y = Rx_{2n}$ and $z = x_{2n+1}$ in condition (1.1), we obtain
$S(fRx_{2n}, fRx_{2n}, gx_{2n+1}) \le q \max \{ S(R^2x_{2n}, R^2x_{2n}, Tx_{2n+1}), S(fRx_{2n}, fRx_{2n}, R^2x_{2n}),$
$S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), (fRx_{2n}, fRx_{2n}, gx_{2n+1})\}$ (1.4)
Now, by taking the upper limit when $n \to\infty$ in (1.5), we get-
$S(Ry, Ry, y) = \lim_{n\to\infty} S(fRx_{2n}, fRx_{2n}, gx_{2n+1})$
$\le q \max \{\lim_{n\to\infty} S(R^2x_{2n}, R^2x_{2n}, Tx_{2n+1}), \lim_{n\to\infty} S(fRx_{2n}, fRx_{2n}, R^2x_{2n}),$
$\lim_{n\to\infty} S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), \lim_{n\to\infty} S(fRx_{2n}, fRx_{2n}, gx_{2n+1})\}$
$\le q \max \{S(Ry, Ry, y), 0, 0, S(Ry, Ry, y)\}$
$= q\, S(Ry, Ry, y)$.
Consequently, $S(Ry, Ry, y) \le q\, S(Ry, Ry, y)$, as $0 < q < 1$ it follows that $Ry = y$.
In a similar way, Since T is continuous, we obtain that
$\lim_{n\to\infty} T^2x_{2n+1} = Ty$, $\lim_{n\to\infty} Tgx_{2n+1} = Ty$.
Since g and T are compatible, $\lim_{n\to\infty} S(gTx_{2n+1}, gTx_{2n+1}, Tgx_{2n+1}) = 0$.

So, by Lemma $\lim_{n\to\infty} gTx_{2n+1} = Ty$.

Putting $x = y = x_{2n}$ and $z = Tx_{2n+1}$ in condition (1.1), we obtain

$$S(fx_{2n}, fx_{2n}, gTx_{2n+1}) \le q \max\{S(Rx_{2n}, Rx_{2n}, T^2x_{2n+1}), S(fx_{2n}, fx_{2n}, Rx_{2n}), S(gTx_{2n+1}, gTx_{2n+1}, T^2x_{2n+1}), S(fx_{2n}, fx_{2n}, gTx_{2n+1})\} \quad (1.5)$$

Similarly, by taking the upper limit when $n \to \infty$ in (1.6), we obtain

$$S(y, y, Ty) = \lim_{n\to\infty} S(fx_{2n}, fx_{2n}, gTx_{2n+1}) \le q \max\{S(y, y, Ty), 0, 0, S(y, y, Ty)\}$$

that is, again it follows that Ty = y.

Also, we can apply condition (1.1) to obtain

$$S(fy, fy, gx_{2n+1}) \le q \max\{S(Ry, Ry, Tx_{2n+1}), S(fy, fy, Ry), S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fy, fy, gx_{2n+1})\} \quad (1.6)$$

And by taking the upper limit when $n \to \infty$ in (1.6), as Ry = Ty= y, we have

$$S(fy, fy, y) \le q \max\{S(Ry, Ry, y), S(fy, fy, y), S(y, y, y), S(fy, fy, y)\} = q\, S(fy, fy, y).$$

Since, $0 < q < 1$, it follows that S(fy, fy, y) = 0 and fy = y.

Finally, by using the condition (1.1) and as, Ry = Ty= fy = y, we obtain

$$S(y, y, gy) = S(fy, fy, gy) \le q \max\{S(Ry, Ry, Ty), S(fy, fy, Ry), S(gy, gy, Ty), S(fy, fy, gy)\} = q\, S(y, y, gy),$$

Which implies that S(y, y, gy) = 0 and gy = y.

Thus, we proved that Ry = Ty = fy = gy = y.

If there exists another common fixed point x in X of all f, g, R, T, then

$$S(x, x, y) = S(fx, fx, gy) \le q \max\{S(Rx, Rx, Ty), S(fx, fx, Rx), S(gy, gy, Ty), S(fx, fx, gy)\} = q \max\{S(x, x, y), S(x, x, x), S(y, y, y), S(x, x, y)\} = q\, S(x, x, y),$$

Which implies that S(x, x, y) = 0 and x = y. thus y is a unique common fixed point of f, g, R and T. The proof of the theorem is completed.

Example-1

Let $X = [0, 1]$ be endowed with S-metric $S(x, y, z) = |x - z| + |y - z|$. Define f, g, R, and T on X by $f(x) = \left(\frac{x}{2}\right)^8$, $g(x) = \left(\frac{x}{2}\right)^4$, $R(x) = \left(\frac{x}{2}\right)^2$, $T(x) = \left(\frac{x}{2}\right)$.

Obviously, $f(X) \subseteq T(X)$ and $g(X) \subseteq R(X)$. Furthermore, the pairs {f, R}, and {g, T} are compatible mappings.Also for each $x, y, z \in X$, we have

$$S(fx, fy, gz) = |fx - gz| + |fy - gz|$$

$$= \left|\left(\frac{x}{2}\right)^8 - \left(\frac{z}{2}\right)^4\right| + \left|\left(\frac{y}{2}\right)^8 - \left(\frac{z}{2}\right)^4\right|$$

$$= \left|\left(\frac{x}{2}\right)^4 - \left(\frac{z}{2}\right)^2\right|\left|\left(\frac{x}{2}\right)^4 + \left(\frac{z}{2}\right)^2\right| + \left|\left(\frac{y}{2}\right)^4 - \left(\frac{z}{2}\right)^2\right|\left|\left(\frac{y}{2}\right)^4 + \left(\frac{z}{2}\right)^2\right|$$

$$\le \frac{5}{16}\left|\left(\frac{x}{2}\right)^2 - \left(\frac{z}{2}\right)\right|\left|\left(\frac{x}{2}\right)^2 + \left(\frac{z}{2}\right)\right| + \frac{5}{16}\left|\left(\frac{y}{2}\right)^2 - \left(\frac{z}{2}\right)\right|\left|\left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)\right|$$

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$$\leq \frac{15}{64}|Rx - Tz| + \frac{15}{64}|Ry - Tz| = \frac{15}{64} S(Rx, Ry, Tz)$$

$$\leq \frac{15}{64} \max\{S(Rx, Ry, Tz), S(fx, fx, Rx), S(gz, gz, Tz), S(fy, fy, gz).$$

Where $\frac{15}{64} \leq q < 1$. Thus f, g, R and T satisfy the conditions given in Theorem and 0 is the unique common fixed point of f, g, R and T.

Corollary 1

Let (X, S) be a complete S-metric space and let f, g : X → X be two mappings such that $S(fx, fy, gz) \leq q \max\{S(x, y, z), S(fx, fx, x), S(gz, gz, z), S(fy, fy, gz)\}$, for all x, y, z ϵ X, with $0 < q < 1$. Then there exists a unique point y ϵ X such that fy = gy= y.

Proof: If we take R and T as identity maps on X, then Theorem (1) follows that f and g have a unique common fixed point.

Theorem 2

Let {f, R} and {g, T} be compatible self-mappings on a complete S-metric space (X, S) and for all x, y, z ϵ X, satisfying

$$S(fx, fy, gz) \leq a_1S(Rx, Ry, Tz) + a_2S(fx, fx, Tz) + a_3S(Rx, Ry, gz) + a_4S(fy, fy, Tz) + a_5S(gz, gz, Tz) \quad (2.1)$$

Where $a_i \geq 0$ (i = 1,2,3,4,5) are real constants with $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$.

If f(X) ⊆ T(X) and g(X) and T are continuous, then all f, g, R and T have a unique common fixed point.

Proof – Let x_0 in X. Since f(X) ⊆ T(X), let x_1 ϵ X be such that $Tx_1 = fx_0$, and also, as gx_1 ϵ R(x), let x_2 ϵ X be such that $Rx_2 = gx_1$.

In general x_{2n+1} ϵ X is chosen such that $Tx_{2n+1} = fx_{2n}$ and x_{2n+2} ϵ X such that $Rx_{2n+2} = gx_{2n+1}$; n = 0, 1, 2, . . .

Denote $y_{2n} = Tx_{2n+1} = fx_{2n}$,
$y_{2n+1} = Rx_{2n+2} = gx_{2n+1}$, $n \geq 0$.

Now, show that $\{y_n\}$ is a Cauchy sequence. For this we have

$$S(y_{2n}, y_{2n}, y_{2n+1}) = S(fx_{2n}, fx_{2n}, gx_{2n+1})$$
$$\leq a_1S(Rx_{2n},Rx_{2n},Tx_{2n+1})+a_2S(fx_{2n},fx_{2n},Tx_{2n+1})+a_3S(Rx_{2n},Rx_{2n},gx_{2n+1}) + a_4 S(fx_{2n}, fx_{2n}, Tx_{2n+1}) + a_5 S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1})$$
$$= a_1S(y_{2n-1},y_{2n-1},y_{2n})+a_2S(y_{2n},y_{2n},y_{2n})+a_3S(y_{2n-1},y_{2n-1},y_{2n+1}) + a_4 S(y_{2n}, y_{2n}, y_{2n}) + a_5 S(y_{2n+1}, y_{2n+1}, y_{2n}).$$
$$\leq a_1S(y_{2n-1},y_{2n-1},y_{2n})+ a_3[2S(y_{2n-1},y_{2n-1},y_{2n}) + S(y_{2n+1}, y_{2n+1}, y_{2n})] + a_5 S(y_{2n}, y_{2n}, y_{2n+1}).$$

Hence, $S(y_{2n}, y_{2n}, y_{2n+1}) \leq a_1S(y_{2n-1},y_{2n-1},y_{2n}) + 2a_3S(y_{2n-1},y_{2n-1},y_{2n}) + (a_3 + a_5)S(y_{2n}, y_{2n}, y_{2n+1}) \quad (2.2)$

Now, we prove that $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1},y_{2n-1},y_{2n})$, for each n ϵ ℕ.

If $S(y_{2n-1},y_{2n-1},y_{2n}) < S(y_{2n}, y_{2n}, y_{2n+1})$ for some n ϵ ℕ, then we have-

$$S(y_{2n}, y_{2n}, y_{2n+1}) < a_1S(y_{2n}, y_{2n}, y_{2n+1}) + 2a_3S(y_{2n}, y_{2n}, y_{2n+1}) + (a_3+a_5) S(y_{2n}, y_{2n}, y_{2n+1})$$
$$= (a_1 + 3a_3 + a_5) S(y_{2n}, y_{2n}, y_{2n+1})$$
$$< S(y_{2n}, y_{2n}, y_{2n+1})$$

which is contradiction. So, we have $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$ and we get $S(y_{2n}, y_{2n}, y_{2n+1}) \leq (a_1 + 3a_3 + a_5)\, S(y_{2n-1}, y_{2n-1}, y_{2n})$ (2.3)

Also, we have

$$\begin{aligned} S(y_{2n-1}, y_{2n-1}, y_{2n}) &= S(y_{2n}, y_{2n}, y_{2n-1}) \\ &= S(fx_{2n}, fx_{2n}, gx_{2n-1}) \\ &\leq a_1S(Rx_{2n}, Rx_{2n}, Tx_{2n-1}) + a_2S(fx_{2n}, fx_{2n}, Tx_{2n-1}) + a_3S(Rx_{2n}, Rx_{2n}, gx_{2n-1}) \\ &\quad + a_4S(fx_{2n}, fx_{2n}, Tx_{2n-1}) + a_5S(gx_{2n-1}, gx_{2n-1}, Tx_{2n-1}) \\ &= a_1S(y_{2n-1}, y_{2n-1}, y_{2n-2}) + a_2S(y_{2n}, y_{2n}, y_{2n-2}) + a_3S(y_{2n-1}, y_{2n-1}, y_{2n-1}) \\ &\quad + a_4S(y_{2n}, y_{2n}, y_{2n-2}) + a_5S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \\ &\leq a_1S(y_{2n-1}, y_{2n-1}, y_{2n-2}) + (2a_2 + 2a_4)S(y_{2n-1}, y_{2n-1}, y_{2n}) + \\ &\quad (a_2 + a_4)S(y_{2n-1}, y_{2n-1}, y_{2n-2}) + a_5S(y_{2n-1}, y_{2n-1}, y_{2n-2}) \end{aligned}$$

Hence, $S(y_{2n}, y_{2n}, y_{2n-1}) \leq a_1S(y_{2n-1}, y_{2n-1}, y_{2n-2}) + (2a_2 + 2a_4)\, S(y_{2n}, y_{2n}, y_{2n-1})$
$+ (a_2 + a_4 + a_5)S(y_{2n-1}, y_{2n-1}, y_{2n-2})$ (2.4)

Similarly, if $S(y_{2n-1}, y_{2n-1}, y_{2n-2}) < S(y_{2n}, y_{2n}, y_{2n-1})$ for some $n \in \mathbb{N}$ then from (2.4) we obtain

$$\begin{aligned} S(y_{2n}, y_{2n}, y_{2n-1}) &\leq (a_1 + 3a_2 + 3a_4 + a_5)\, S(y_{2n}, y_{2n}, y_{2n-1}) \\ &< S(y_{2n}, y_{2n}, y_{2n-1}) \end{aligned}$$

which is contradiction.

So, we have $S(y_{2n}, y_{2n}, y_{2n-1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n-2})$, for each $n \in \mathbb{N}$ then from (2.4) we get $S(y_{2n}, y_{2n}, y_{2n-1}) \leq (a_1 + 3a_2 + 3a_4 + a_5)\, S(y_{2n-1}, y_{2n-1}, y_{2n-2})$ (2.5)

Now, from (2.3) and (2.5) we have,

$S(y_n, y_n, y_{n-1}) < \lambda\, S(y_{n-1}, y_{n-1}, y_{n-2}),\ n \geq 2,$

Where $\lambda = \min\{a_1 + 3a_3 + a_5, a_1 + 3a_2 + 3a_4 + a_5)$. We know that $\lambda \in (0, 1)$.

Hence, for $n \geq 2$ it follows that

$S(y_n, y_n, y_{n-1}) \leq \ldots \leq \lambda^{n-1} S(y_1, y_1, y_0).$ (2.6)

by the triangle inequality in S-metric space, for $n > m$ we have

$S(y_n, y_n, y_m) \leq 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) + \ldots + 2S(y_{n-1}, y_{n-1}, y_n)$

Hence, from (2.6) and as $\lambda < 1$, we have

$$\begin{aligned} S(y_n, y_n, y_m) &\leq 2(\lambda^m + \lambda^{m+1} + \ldots + \lambda^{n-1})\, S(y_1, y_1, y_0) \\ &\leq 2\lambda^m [1 + \lambda + (\lambda)^2 + \ldots]\, S(y_1, y_1, y_0) \\ &\leq \frac{2\lambda^m}{1-\lambda} S(y_1, y_1, y_0) = \frac{2\lambda^m}{1-\lambda} S(y_1, y_1, y_0) \to 0 \text{ as } m \to \infty. \end{aligned}$$

It follows that, $\{y_n\}$ is a Cauchy sequence. Let $y \in X$ be such that

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Rx_{2n+2} = y$$

Since, R is continuous it follows that $\lim_{n\to\infty} R^2x_{2n+2} = Ry$, $\lim_{n\to\infty} Rfx_{2n} = Ry$.

And since, f and R are compatible, $\lim_{n\to\infty} S(fRx_{2n}, fRx_{2n}, Rfx_{2n}) = 0$.

So, by Lemma, $\lim_{n\to\infty} fRx_{2n} = Ry$. From, (1.8) it follows that

$$\begin{aligned} S(fRx_{2n}, fRx_{2n}, gx_{2n+1}) &\leq a_1S(R^2x_{2n}, R^2x_{2n}, Tx_{2n+1}) + a_2S(fRx_{2n}, fRx_{2n}, Tx_{2n+1}) \\ &\quad + a_3S(R^2x_{2n}, R^2x_{2n}, gx_{2n+1}) + a_4S(fRx_{2n}, fRx_{2n}, Tx_{2n+1}) \\ &\quad + a_5 S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}) \end{aligned}$$

Taking the upper limit as $n \to \infty$, we get

$S(Ry, Ry, y) \leq a_1S(Ry, Ry, y) + a_2S(Ry, Ry, y) + a_3S(Ry, Ry, y) + a_4S(Ry, Ry, y) +$

$a_5\, S(y, y, y)$
$\leq a_1\, S(Ry, Ry, y) + a_2\, S(Ry, Ry, y) + a_3 S(Ry, Ry, y) + a_4 S(Ry, Ry, y)$
$= (a_1 + a_2 + a_3 + a_4) S(Ry, Ry, y)$
$\leq (a_1 + a_2 + a_3 + a_4 + a_5) S(Ry, Ry, y)$

Therefore, $S(Ry, Ry, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(Ry, Ry, y)$,
as $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, we know that $Ry = y$.
In a similar way, since T is continuous, we obtain that $\lim_{n\to\infty} T^2 x_{2n+1} = Ty$,
$\lim_{n\to\infty} Tgx_{2n+1} = Ty$.
Since g and T are compatible, $\lim_{n\to\infty} S(gTx_{2n+1}, gTx_{2n+1}, Tgx_{2n+1}) = 0$.
So, by Lemma, $\lim_{n\to\infty} gTx_{2n+1} = Ty$.
From (1.8) it follows that

$S(fx_{2n}, fx_{2n}, gT_{2n+1}) \leq a_1 S(Rx_{2n}, Rx_{2n}, T^2 x_{2n+1}) + a_2 S(fx_{2n}, fx_{2n}, T^2 x_{2n+1})$
$+ a_3 S(Rx_{2n}, Rx_{2n}, gTx_{2n+1}) + a_4 S(fx_{2n}, fx_{2n}, T^2 x_{2n+1})$
$+ a_5\, S(gTx_{2n+1}, gTx_{2n+1}, T^2 x_{2n+1})$

Taking the upper limit as $n \to \infty$, we get

$S(y, y, Ty) \leq a_1 S(y, y, Ty) + a_2 S(y, y, Ty) + a_3 S(y, y, Ty) + a_4 S(y, y, Ty) + a_5 S(Ty, Ty, Ty)$
$= (a_1 + a_2 + a_3 + a_4) S(Ty, Ty, y)$
$\leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(y, y, Ty)$,

that is, $S(y, y, Ty) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(y, y, Ty)$
Therefore, by $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, we know that $Ty = y$.
Again from (1.8), it follows that

$S(fy, fy, gx_{2n+1}) \leq a_1 S(Ry, Ry, Tx_{2n+1}) + a_2 S(fy, fy, Tx_{2n+1})$
$+ a_3 S(Ry, Ry, gx_{2n+1}) + a_4 S(fy, fy, Tx_{2n+1}) + a_5 S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1})$

And by taking the upper limit as $n \to \infty$, as $Ry = y$, $Ty = y$, we get

$S(fy, fy, y) \leq a_1 S(y, y, y) + a_2 S(fy, fy, y) + a_3 S(y, y, y) + a_4 S(fy, fy, y) + a_5 S(y, y, y)$
$\leq (a_2 + a_4)\, S(fy, fy, y)$

Therefore, $S(fy, fy, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(fy, fy, y)$ and
by $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, we know that $fy = y$.
Again from (1.8), we have $S(fy, fy, gy) = 0$; hence $fy = gy$. Thus we prove that $fy = gy = Ry = Ty = y$, if there exists another common fixed point x in X of all f, g, R, T, then

$S(x, x, y) = S(fx, fx, gy)$
$\leq a_1 S(Rx, Rx, Ty) + a_2 S(fx, fx, Ty) + a_3 S(Rx, Rx, gy) + a_4 S(fx, fx, Ty) +$
$a_5 S(gy, gy, Ty)$
$= (a_1 + a_2 + a_3 + a_4) S(x, x, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(x, x, y)$

From which it follows, $S(x, x, y) \leq (a_1 + 3a_2 + 3a_3 + 3a_4 + a_5) S(x, x, y)$.
Since, $a_1 + 3a_2 + 3a_3 + 3a_4 + a_5 < 1$, it follows that $S(x, x, y) = 0$, i.e., $x = y$. Therefore, y is a unique common fixed point of all f, g, R, T. The proof of the Theorem is completed.

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