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A Study on Fixed Point Results within Fuzzy Metric Structures

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ABSTRACT
Fixed point theory provides a rigorous mechanism for proving the existence and uniqueness of equilibrium states and for validating iterative approximation procedures. Fuzzy metric structures extend this mechanism to settings in which nearness is represented by a graded value that depends on both the pair of points and a positive scale parameter. This article presents a theoretical study of fixed point results in George–Veeramani fuzzy metric spaces. A reciprocal-gap functional is used to convert fuzzy nearness into a non-negative separation quantity. On this basis, an orbit-bounded contraction principle is formulated and proved. The theorem establishes that a reciprocal-gap contractive self-map on a complete fuzzy metric space possesses a unique fixed point whenever one Picard orbit is bounded at every positive scale. The argument gives an explicit Cauchy estimate and does not require an independent continuity assumption on the mapping. A metric-induced corollary recovers the classical Banach contraction principle, while a commuting-map corollary yields a unique common fixed point. A fully worked example on the non-negative real line illustrates the hypotheses, iteration formula, rate of convergence, and limiting fuzzy nearness. The study clarifies the relationship between ordinary and fuzzy contraction mechanisms and provides a transparent framework for further nonlinear extensions.
Keywords: fixed point; fuzzy metric space; continuous triangular norm; reciprocal-gap contraction; Picard iteration; completeness; common fixed point

1. INTRODUCTION

Fixed point theory studies conditions under which a mapping returns at least one point to itself. The classical contraction principle of Banach gives existence, uniqueness, and convergence of successive approximations in a complete metric space [1]. Its importance lies not only in the assertion that a solution exists, but also in the constructive sequence generated by repeated

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application of the mapping. When uncertainty, imprecision, or graded similarity is intrinsic to the model, an ordinary distance may be too rigid. Zadeh's formulation of fuzzy sets supplied the membership-based language needed to represent such gradation [2].

The probabilistic foundations of generalized distance were developed through statistical metric spaces, where distance information is described by distribution functions rather than a single non-negative number [3]. Kramosil and Michálek introduced fuzzy metrics as a related structure and connected fuzzy nearness with statistical metric ideas [4]. George and Veeramani subsequently modified the axioms so that a fuzzy metric generates a well-behaved Hausdorff topology and admits a natural theory of convergence and completeness [5]. This formulation is now one of the principal settings for fixed point analysis in fuzzy spaces.

The first influential contraction theorem in a fuzzy metric framework was developed by Grabiec, who used a scale-changing contractive condition and a strong form of completeness [6]. Earlier probabilistic contraction results of Sehgal and Bharucha-Reid also showed how iterative fixed-point methods can operate when distance is distribution-valued [7]. Gregori and Sapena proposed a reciprocal expression involving $1/M-1$, providing a particularly direct fuzzy analogue of a linear metric contraction [8]. Miheţ established a Banach-type theorem under alternative completeness and contraction requirements [9] and later developed ψ-contractive mappings for non-Archimedean fuzzy metric spaces [10].

Further work has examined the precise strength of fuzzy contractive definitions. Gregori and Miñana identified important distinctions among contraction concepts and clarified the assumptions needed in fixed point arguments [11]. Their later treatment of fuzzy ψ-contractive sequences emphasized that the behavior of the entire iterative sequence is as important as the pointwise inequality imposed on the mapping [12]. Wardowski introduced fuzzy contractive mappings based on a more flexible transforming function [13], while his F-contraction framework in ordinary complete metric spaces stimulated many related nonlinear contractive schemes [14]. Shukla, Gopal, and Sintunavarat developed another broad class of fuzzy contractions and corresponding fixed-point results [15].

Modern investigations increasingly focus on completeness, contractive sequences, and the structural relation between a fuzzy metric and its underlying topology. The contractive-sequence approach of Gregori, Miñana, and Miravet makes the Cauchy stage of a proof explicit [16]. Work on completable fuzzy metric spaces explains when an incomplete fuzzy structure can be embedded into a complete one [17], and Hausdorff fuzzy metrics extend fuzzy nearness to compact subsets [18]. Common fixed-point results, including those of Sharma, show how compatibility conditions can coordinate several self-maps [19]. Recent surveys confirm that simulation functions, generalized contractions, proximal conditions, and nonlinear applications remain active directions of research [20].

Against this background, the present article isolates a simple but useful mechanism: measure separation through the reciprocal gap $\Delta_t(x,y)=1/M(x,y,t)-1$, impose a uniform linear decrease on that quantity, and require one Picard orbit to be bounded at every scale. This

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combination yields an elementary and transparent existence–uniqueness theorem in a complete fuzzy metric space. The result also shows exactly how the classical Banach estimate is recovered when the fuzzy metric is induced by an ordinary metric.

2. OBJECTIVES OF THE STUDY

The objectives of this theoretical investigation are:

1. To present the essential definitions of continuous triangular norms, fuzzy metrics, convergence, Cauchy sequences, and completeness;
2. To express fuzzy contractivity through a reciprocal-gap functional that behaves like a scale-dependent distance;
3. To establish an orbit-bounded fixed-point theorem with complete proof of existence and uniqueness;
4. To connect the theorem with the classical Banach contraction principle and a common fixed-point result; and
5. To demonstrate the iterative mechanism through a solved example.

3. RESEARCH METHODOLOGY

The study follows a deductive mathematical method. First, the George–Veeramani axioms are adopted as the ambient fuzzy metric structure. Second, a real-valued reciprocal-gap functional is associated with the fuzzy metric at every fixed scale $t > 0$. Third, contractive inequality is iterated along with the Picard sequence $x_{n+1} = Tx_n$. The bounded-orbit hypothesis is then used to obtain a uniform Cauchy estimate for all indices $m \geq n$. Completeness supplies a limit, and the same contractive inequality identifies the limit as a fixed point. Finally, uniqueness is proved directly, and two corollaries and an example are derived. No empirical data, sampling procedure, or statistical test is involved because the article is pure theoretical analysis.

4. PRELIMINARIES

4.1 Continuous Triangular Norm

A **continuous triangular norm**, or continuous t-norm, is a mapping

$$*: [0,1] \times [0,1] \to [0,1] \tag{4.1}$$

that is commutative, associative, continuous, non-decreasing in each argument, and satisfies

$$a * 1 = a \qquad \text{for every } a \in [0,1]. \tag{4.2}$$

Typical examples are the minimum t-norm $a * b = \min\{a, b\}$, the product t-norm $a * b = ab$, and the Łukasiewicz t-norm $a * b = \max\{a + b - 1,0\}$.

4.2 Fuzzy Metric Space

Let X be a non-empty set and let $*$ be a continuous t-norm. A mapping

$$M: X \times X \times (0, \infty) \to (0,1] \tag{4.3}$$

is called a fuzzy metric if, for all $x, y, z \in X$ and $s, t > 0$, the following conditions hold:

1. $M(x, y, t) > 0$;
2. $M(x, y, t) = 1$ if and only if $x = y$;
3. $M(x, y, t) = M(y, x, t)$;
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$; and
5. the function $t \mapsto M(x, y, t)$ is continuous.

The triple $(X, M,*)$ is then called a **fuzzy metric space**. The value $M(x, y, t)$ represents the degree of nearness of x and y at scale t. Larger values indicate greater nearness, and the value 1 characterizes equality.

4.3 Convergence, Cauchy Property, and Completeness

A sequence $\{x_n\}$ converges to $x \in X$ when

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \qquad \text{for every } t > 0. \tag{4.4}$$

It is Cauchy when, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $N \in \mathbb{N}$ such that

$$M(x_m, x_n, t) > 1 - \varepsilon \qquad \text{whenever } m, n \geq N. \tag{4.5}$$

The fuzzy metric space is complete if each Cauchy sequence converges to a point of X.

4.4 Reciprocal-Gap Functional

For each $t > 0$, define

$$\Delta_t(x, y) = \frac{1}{M(x, y, t)} - 1. \tag{4.6}$$

Because $0 < M(x, y, t) \leq 1$, the quantity $\Delta_t(x, y)$ is non-negative, and

$$\Delta_t(x, y) = 0 \quad \Leftrightarrow \quad x = y. \tag{4.7}$$

Moreover, $M(x_n, x, t) \to 1$ if and only if $\Delta_t(x_n, x) \to 0$. Thus, Δ_t translates fuzzy nearness into an ordinary non-negative separation quantity without replacing the underlying fuzzy metric.

4.5 Uniqueness of Fuzzy Limits

Suppose $x_n \to p$ and $x_n \to q$. For any $t > 0$, the fuzzy triangle inequality gives

$$M(p,q,t) \geq M\left(p, x_n, \frac{t}{2}\right) * M\left(x_n, q, \frac{t}{2}\right). \tag{4.8}$$

Letting $n \to \infty$, continuity of the t-norm yields $M(p,q,t) \geq 1$. Hence $M(p,q,t) = 1$, so $p = q$. Therefore, a convergent sequence in a fuzzy metric space has a unique limit.

5. MAIN FIXED-POINT RESULT

5.1 Reciprocal-Gap Contraction

Let $(X, M,*)$ be a fuzzy metric space. A self-map $T: X \to X$ is called a reciprocal-gap contraction if there exists a constant $\lambda \in [0,1)$ such that

$$\Delta_t(Tx, Ty) \leq \lambda\, \Delta_t(x,y) \qquad (x, y \in X,\ t > 0). \tag{5.1}$$

Equivalently,

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq \lambda\left(\frac{1}{M(x,y,t)} - 1\right). \tag{5.2}$$

Repeated application immediately gives the iterative estimate

$$\Delta_t(T^n x, T^n y) \leq \lambda^n \Delta_t(x,y) \qquad (n \in \mathbb{N}). \tag{5.3}$$

5.2 Orbit-Bounded Fixed-Point Theorem

Theorem 1. Let $(X, M,*)$ be a complete fuzzy metric space and let $T: X \to X$ be a reciprocal-gap contraction with constant $\lambda \in [0,1)$. Choose $x_0 \in X$ and define $x_n = T^n x_0$. Assume that, for every $t > 0$, the orbit is t-wise reciprocal-gap bounded; that is,

$$B_t(x_0) := \sup_{k \geq 0} \Delta_t(x_k, x_0) < \infty. \tag{5.4}$$

Then T has a unique fixed point $p \in X$, and the Picard sequence $\{x_n\}$ converges to p.

Proof. Let $m \geq n$. Since $x_m = T^n x_{m-n}$ and $x_n = T^n x_0$, the iterated contraction inequality implies

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$$\Delta_t(x_m, x_n) \le \lambda^n \Delta_t(x_{m-n}, x_0) \le \lambda^n B_t(x_0). \tag{5.5}$$

For fixed $t > 0$, the right-hand side tends to zero as $n \to \infty$, uniformly for all $m \ge n$. Hence

$$\lim_{m,n\to\infty} \Delta_t(x_m, x_n) = 0, \tag{5.6}$$

or equivalently,

$$\lim_{m,n\to\infty} M(x_m, x_n, t) = 1. \tag{5.7}$$

Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X provides a point $p \in X$ such that $x_n \to p$.
It remains to show that p is fixed. Using $x_{n+1} = Tx_n$, one obtains

$$\Delta_t(x_{n+1}, Tp) = \Delta_t(Tx_n, Tp) \le \lambda\, \Delta_t(x_n, p). \tag{5.8}$$

The right-hand side tends to zero, so $x_{n+1} \to Tp$. The tail sequence $\{x_{n+1}\}$ also converges to p. By uniqueness of fuzzy limits, $Tp = p$.
For uniqueness, suppose that p and q are fixed points. Then

$$\Delta_t(p, q) = \Delta_t(Tp, Tq) \le \lambda\, \Delta_t(p, q). \tag{5.9}$$

Since $1 - \lambda > 0$, this inequality forces $\Delta_t(p, q) = 0$, and therefore $p = q$. The fixed point is unique. ▫

5.3 Interpretation of Assumptions

The contraction controls how much reciprocal separation remains after one iteration. The orbit-bounded condition prevents the comparison term $\Delta_t(x_{m-n}, x_0)$ from growing rapidly enough to cancel the decay of λ^n. This condition is weaker than requiring the whole fuzzy metric space to be bounded; only one orbit is needed. The proof also shows that a separate continuity hypothesis on T is unnecessary, because the contraction itself gives sequential continuity:

$$x_n \to x \quad \Rightarrow \quad \Delta_t(Tx_n, Tx) \le \lambda \Delta_t(x_n, x) \to 0. \tag{5.10}$$

6. CONSEQUENCES AND EXTENSIONS

6.1 Recovery of the Banach Contraction Principle

Let (X, d) be a complete metric space and define the standard fuzzy metric

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \tag{6.1}$$

with the product t-norm. Its reciprocal gap is

$$\Delta_t(x, y) = \frac{d(x, y)}{t}. \tag{6.2}$$

If T satisfies the ordinary contraction inequality

$$d(Tx, Ty) \le \lambda d(x, y), \qquad 0 \le \lambda < 1, \tag{6.3}$$

then it is a reciprocal-gap contraction for M_d. In addition,

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$$d(x_k, x_0) \le \sum_{j=0}^{k-1} d\left(x_{j+1}, x_j\right) \le \frac{d(x_1, x_0)}{1-\lambda}, \tag{6.4}$$

So the required orbit bound holds for every $t > 0$. Theorem 1 therefore gives the unique Banach fixed point. The familiar error estimate becomes

$$d(x_n, p) \le \frac{\lambda^n}{1-\lambda} d(x_1, x_0), \tag{6.5}$$

and, in fuzzy form,

$$M_d(x_n, p, t) \ge \frac{t}{t + \frac{\lambda^n}{1-\lambda} d(x_1, x_0)}. \tag{6.6}$$

This shows that the fuzzy conclusion contains not merely qualitative convergence but also a quantitative lower bound for nearness to the fixed point.

6.2 A Common Fixed-Point Corollary

Corollary 1. Let $S, T: X \to X$ be commuting self-maps on a completely fuzzy metric space. Suppose each map satisfies the hypotheses of Theorem 1, possibly with different contraction constants and different bounded initial orbits. Then S and T have a unique common fixed point.

Proof. Theorem 1 gives a unique fixed point p of S. Since $ST = TS$,

$$S(Tp) = T(Sp) = Tp. \tag{6.7}$$

Thus Tp is also a fixed point of S. Uniqueness for S gives $Tp = p$, so p is common to both maps. The unique fixed point of T must therefore be p. ▫

7. ILLUSTRATIVE EXAMPLE

Let $X = [0, \infty)$ with the usual distance $d(x, y) = |x - y|$, and let

$$M(x, y, t) = \frac{t}{t + |x - y|} \tag{7.1}$$

with the product t-norm. Define

$$T(x) = \frac{x+2}{3}. \tag{7.2}$$

Then

$$\Delta_t(Tx, Ty) = \frac{|Tx - Ty|}{t} = \frac{1}{3}\frac{|x-y|}{t} = \frac{1}{3}\Delta_t(x, y). \tag{7.3}$$

Hence T is a reciprocal-gap contraction with $\lambda = 1/3$. Its iterates are

$$x_n = T^n x_0 = 1 + \frac{x_0 - 1}{3^n}. \tag{7.4}$$

Moreover,

$$\Delta_t(x_k, x_0) = \frac{|x_k - x_0|}{t} = \frac{|x_0 - 1|\,(1 - 3^{-k})}{t} \le \frac{|x_0 - 1|}{t}, \tag{7.5}$$

So the orbit-bounded hypothesis is satisfied. Theorem 1 gives the unique fixed point $p = 1$.

For example, from $x_0 = 7$, the iteration produces $x_1 = 3$, $x_2 = 5/3$, and $x_3 = 11/9$. The next two values are $x_4 = 29/27$ and $x_5 = 83/81$. At scale $t = 1$, the nearness to the fixed point is

$$M(x_n, 1,1) = \frac{1}{1 + 6/3^n}, \tag{7.6}$$

which increases from $1/7$ at $n = 0$ to $1/3$, $3/5$, $9/11$, $27/29$, and $81/83$. The values approach 1, exactly as fuzzy convergence requires.

8. SIGNIFICANCE AND SCOPE

Reciprocal-gap formulation has three advantages. First, it converts the direction of fuzzy nearness into the familiar direction of distance reduction: smaller Δ_t means greater nearness. Second, its iterative estimate is immediate and produces a direct Cauchy bound. Third, the standard fuzzy metric reduces the entire argument to the ordinary Banach theory through the identity $\Delta_t = d/t$.

The theorem is intentionally focused. The orbit-bounded hypothesis is sufficient and easily verified in metric-induced examples, but it is not claimed to be necessary. Future extensions may replace the linear factor λ by a comparison function, allow cyclic or multivalued operators, weaken commutativity through compatibility conditions, or study best proximity points when a true fixed point cannot exist. The same framework may also be adapted to integral equations, iterative algorithms, and equilibrium models after the operator and the fuzzy metric are specified.

9. CONCLUSION

This article has presented a concise fixed-point framework within completely fuzzy metric structures. The reciprocal-gap functional $\Delta_t = 1/M - 1$ provides an interpretable bridge between fuzzy nearness and ordinary separation. Under a linear reciprocal-gap contraction and boundedness of one Picard orbit at each positive scale, the iterates form a Cauchy sequence, converge to a fixed point, and determine that point uniquely. The contraction itself is sufficient to pass the limit through the mapping, so an additional continuity assumption is not required.

The metric-induced corollary recovers the Banach contraction principle together with an explicit fuzzy convergence estimate, and the commuting-map corollary supplies a unique common fixed point. The worked example confirms every hypothesis and displays the increasing degree of nearness along the iteration. Consequently, the approach offers a clear theoretical template for analyzing more general fuzzy contractions while preserving the constructive strength of classical fixed-point theory.

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